Lyapunov-type inequalities for a class of nonlinear higher order differential equations

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Abstract

In this work, we establish several Lyapunov-type inequalities for a class of nonlinear higher order differential equations having a form

$$(\psi(u^{(m)}(x)))' + \sum_{i=0}^{n} r_i(x) f_i(u^{(i)}(x)) = 0,$$

with anti-periodic boundary conditions, where $m > n \geq 0$ are integers, ψ and $f_i (i = 0, 1, 2, ..., n)$ satisfy certain structural conditions such that the considered equations have general nonlinearities. The obtained inequalities are extensions and complements of the existing results in the literature.

Key words: Lyapunov inequality, Higher order differential equation, nonlinear equation.

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1 Introduction

If u is a nontrivial solution of the following second order differential equation

$$u''(x) + r(x)u(x) = 0, \ \ x \in (a, b),$$
$$u(a) = u(b) = 0,$$

where r is a continuous and nonnegative function defined in $[a, b]$, then the following inequality

$$\int_a^b r(x)\mathrm{d}x \geq \frac{4}{b-a}, \tag{1}$$

holds, which was first proved by Lyapunov [7] and known as "Lyapunov inequality". The Lyapunov inequality and its various generalizations have proved to be useful tools in oscillation theory, disconjugacy, eigenvalue problems, and numerous other applications for the theories of differential and difference equations, and also in time scales. Since the appearance of Lyapunov's fundamental paper, there are many improvements and generalizations of (1) in the literature. Especially, the Lyapunov inequality has been generalized extensively to the higher-order linear equations see [1–4,6,8–13,15,14,16,18] and references therein. But so far, only a few results have been achieved for higher order differential equations that have either general nonlinearities or anti-periodic boundary conditions. For example, [12] and [17] have contributed to certain higher equations with anti-periodic boundary conditions, and to general nonlinear second order equations with Dirichlet boundary conditions respectively.

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In 2012, Wang [12] considered Lyapunov-type inequalities for certain higher order differential equations with anti-periodic boundary conditions. The main result in [12] is as follows:
Theorem A If $u(x)$ is a nonzero solution of the following $(m+1)$-order half-linear differential equation anti-periodic boundary conditions:

$$
\begin{aligned}
&(|u^{(m)}(x)|^{p-2}u^{(m)}(x)) + r(x)|u(x)|^{p-2}u(x) = 0, \ \ x \in (a,b),\\
&u^{(i)}(a) + u^{(i)}(b) = 0, i = 0,1,2,...,m,
\end{aligned}
$$

then

$$
\int_a^b |r(x)|dx > 2\left(\frac{2}{b-a}\right)^{m(p-1)}. \tag{2}
$$

In a recent work [17], the authors established Lyapunov-type inequalities for a class of nonlinear second order equations with a general form as

$$
(\psi(u'(x)))' + r(x)f(u(x)) = 0, \ \ x \in (a,b), \tag{3a}
$$

$$
u(a) = u(b) = 0. \tag{3b}
$$

The structural conditions are given as below, which allow the functions ψ and f to have a large class of nonlinearities (see Section 1 and 4 in [17], and Remark 1, for instance):

(A) $\psi, f \in C((-\infty,\infty)) \cap C^1((0,\infty))$ with $f \not\equiv 0$ on $(-\infty,\infty)$.
(B) ψ is odd on $(-\infty,\infty)$.
(C) $f(t) \geq 0$ for all $t \in [0,\infty)$.
(D) There exists $k_0 > 0$ such that $|f(t)| \leq k_0\psi(|t|)$ for all $t \in (-\infty,\infty)$.
(E) There exists constants $\alpha, \beta \geq 0$ such that

$$
\alpha\psi(t) \leq t\psi'(t) \leq \beta\psi(t), \ \ \forall t > 0.
$$

(F) There exists constants $\theta_0, \delta_0 \geq 0$ such that

$$
\theta_0 f(t) \leq tf'(t) \leq \delta_0 f(t), \ \ \forall t > 0.
$$

The main result in [17] is the following theorem.
Theorem B Suppose that conditions (A)-(D) are satisfied. Let u be a nontrivial solution of (3).

(i) If ψ satisfies (E), then $\int_a^b |r(x)|\mathrm{d}x \geq \frac{2}{k_0} \cdot \frac{1+\alpha}{1+\beta} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{\alpha}, \left(\frac{2}{b-a}\right)^{\beta}\right\}$.
(ii) If f satisfies (F), then $\int_a^b |r(x)|\mathrm{d}x \geq \frac{2}{k_0} \cdot \frac{1+\theta_0}{1+\delta_0} \cdot \min\left\{\left(\frac{2}{b-a}\right)^{\theta_0}, \left(\frac{2}{b-a}\right)^{\delta_0}\right\}$.

Our motivation for this paper comes from the papers of [12] and [17]. We attempt to establish several new Lyapunov-type inequalities for a large class of nonlinear higher differential equations under anti-periodic boundary conditions. The results obtained in this paper are generalizations of [12] and [17].

2 Problem setting and main results

In this paper, we establish Lyapunov inequalities for the following equation

$$
(\psi(u^{(m)}(x)))' + \sum_{i=0}^{n} r_i(x)f_i(u^{(i)}(x)) = 0, \ \ x \in (a,b), \tag{4a}
$$

$$
u^{(j)}(a) + u^{(j)}(b) = 0, \ \ j = 0,1,2,...,m, \tag{4b}
$$

where r is a given function defined in $[a,b]$, ψ and f satisfy some of the following structural conditions:

(H_1) $\psi, f \in C((-\infty,\infty)) \cap C^1((0,\infty))$ with $\sum\limits_{i=0}^{n} |f_i| \not\equiv 0$ on $(-\infty,\infty)$, $i = 0, 1, 2, ..., n$.

(H_2) ψ is odd on $(-\infty,\infty)$.

(H_3) $f_i(t) \geq 0$ for all $t \in [0,\infty)$, $i = 0, 1, 2, ..., n$.

(H_4) There exists constants $k_i \geq 0$ satisfying $\sum\limits_{i=0}^{n} k_i > 0$ such that $|f_i(t)| \leq k_i \psi(|t|)$ for all $t \in (-\infty,\infty)$, $i = 0, 1, 2, ..., n$.

(H_ψ) There exists constants $\alpha, \beta \geq 0$ such that

$$\alpha\psi(t) \leq t\psi'(t) \leq \beta\psi(t), \quad \forall t > 0.$$

(H_{f_i}) There exists constants $\theta_i, \delta_i \geq 0$ such that

$$\theta_i f_i(t) \leq t f_i'(t) \leq \delta_i f_i(t), \quad \forall t > 0, i = 0, 1, 2, ..., n.$$

Remark 1 *(i) The structural condition (H_ψ) (or (H_{f_i})) was firstly presented in [17], which is a slight version given by Lieberman [5]. Observe that $\alpha = \beta = p - 1$ when $\psi(t) = |t|^{p-2}t$ $(p > 1)$, and $\alpha = a, \beta = a + \frac{1}{\ln d}$ in (H_ψ) when $\psi(t) = |t|^{a-1}t\log_c(b|t| + d)$, $a, b > 0, c, d > 1$. Another interesting example of ψ is $\psi(t) = \frac{|t|^{a-1}t}{\log_c(b|t|+d)}$, $b > 0, c, d > 1, a > \frac{1}{\ln d}$ with $\alpha = a - \frac{1}{\ln d}, \beta = a$ in (H_ψ). More examples of nonlinear functions satisfying (H_ψ) can be found in [17, Section 1 and 4].*

(ii) By $(H_1) - (H_4)$, $\psi(0) = f_i(0) = 0$ $(i = 0, 1, 2, ..., n)$ and $\psi(t) \geq 0$ for any $t \geq 0$. Furthermore, if ψ (or f_i) satisfies (H_ψ) (or (H_{f_i})), then $\psi'(t) \geq 0$ (or $f_i'(t) \geq 0$), which guarantees the increasing monotonicity of $\psi(t)$ (or $f_i(t)$) in $t \geq 0$.

Throughout this paper, we always assume that $m > n \geq 0$ are integers, $r_i \in L^1(a, b)$ with $r_i \not\equiv 0$ on (a, b) $(i = 0, 1, 2, ..., n)$, and conditions $(H_1) - (H_4)$ are satisfied. Moreover, we say u is a solution of (4) if $u \in C^{m+1}(a, b) \cap C^m([a, b])$, $\psi(u^{(m)}(x))$ is absolutely continuous in x, and u satisfies the equation in (4) almost everywhere in (a, b).

The main result in this paper is given below.

Theorem 1 *Suppose that* (4) *admits a nontrivial solution.*

(i) If ψ satisfies (H_ψ), then $\sum\limits_{i=0}^{n} \left(\int_a^b |r_i| dx \cdot k_i \cdot \max\left\{ \left(\frac{b-a}{2}\right)^{\alpha(m-i)}, \left(\frac{b-a}{2}\right)^{\beta(m-i)} \right\} \right) \geq \frac{2(1+\alpha)}{1+\beta}$.

(ii) If f_i satisfies (H_{f_i}) $(i = 0, 1, 2, ..., n)$, then $\sum\limits_{i=0}^{n} \left(\int_a^b |r_i| dx \cdot k_i \cdot \frac{1+\delta_i}{1+\theta_i} \cdot \max\left\{ \left(\frac{b-a}{2}\right)^{\theta_i(m-i)}, \left(\frac{b-a}{2}\right)^{\delta_i(m-i)} \right\} \right) \geq 2.$

Corollary 2 *If u is a nontrivial solution of* (4) *with $\psi(t) = f_i(t) = |t|^{p-2}t, p > 1, i = 0, 1, 2, ..., n$, then*

$$\sum_{i=0}^{n} \left(\int_a^b |r_i| dx \cdot \left(\frac{b-a}{2}\right)^{(p-1)(m-i)} \right) \geq 2.$$

Particulary, if $n = 0$, there holds $\int_a^b |r_0| dx \geq \frac{2^{m(p-1)+1}}{(b-a)^{m(p-1)}}$, which is the same as (2).

3 Proof of main result

We present first some auxiliary results needed in the main proofs. In this section, we always write $\Psi(t) = \int_0^t \psi(s)\mathrm{d}s$ and $F_i(t) = \int_0^t f_i(s)\mathrm{d}s$ $(i = 0, 1, 2, ..., n)$ for $t \geq 0$.

Lemma 3 *[17] Assume that ψ satisfies (H_1)-(H_4) and (H_ψ). The following results hold true.*

(i) $\psi(st) \leq \max\{s^\alpha, s^\beta\}\psi(t)$, $\forall s, t \geq 0$.

(ii) Ψ *is C^2-contiuous on $(0, +\infty)$, and convex on $[0, +\infty)$.*

(iii) $\frac{t\psi(t)}{1+\beta} \leq \Psi(t) \leq \frac{t\psi(t)}{1+\alpha}$, $\forall t \geq 0$.

Remark 2 *The function f_i satisfying (H_1)-(H_4) and (H_{f_i}) and the function F_i have similar properties as above.*

Lemma 4 *Let u be a C^m-contious function on $[a,b]$ satisfying* (4b). *The following inequalities hold:*

$$\max_{x\in[a,b]}|u^{(i)}(x)|\leq\left(\frac{b-a}{2}\right)^{m-i}\cdot\frac{1}{b-a}\cdot\int_a^b|u^{(m)}(x)|dx,\quad i=0,1,2,..,m-1.$$

Proof. It suffices to note that

$$\begin{aligned}|u^{(i)}(x)|&=\frac{1}{2}\Big|\int_a^x u^{(i+1)}(y)\mathrm{d}y+\int_b^x u^{(i+1)}(y)\mathrm{d}y\Big|\\&\leq\frac{1}{2}\Big(\int_a^x|u^{(i+1)}(y)|\mathrm{d}y+\int_x^b|u^{(i+1)}(y)|\mathrm{d}y\Big)\\&\leq\frac{1}{2}\int_a^b|u^{(i+1)}(y)|\mathrm{d}y\ \ (i=0,1,2,..,m-1).\end{aligned}$$

■

Lemma 5 *Let u be a C^m-continuous function on $[a,b]$ satisfying* (4b).

(i) If f_i satisfies (H_{f_i}) $(i=0,1,2,..,n)$, then

$$\begin{aligned}&f_i\left(\left(\frac{b-a}{2}\right)^{m-i}\cdot\frac{1}{b-a}\cdot\int_a^b|u^{(m)}(x)|dx\right)\cdot\int_a^b|u^{(m)}(x)|dx\\&\leq(1+\delta_i)\cdot\max\left\{\left(\frac{b-a}{2}\right)^{\theta_i(m-i)},\left(\frac{b-a}{2}\right)^{\delta_i(m-i)}\right\}\cdot\int_a^b F_i(|u^{(m)}(x)|)dx.\end{aligned}$$

(ii) If ψ satisfies (H_ψ), then for $i=0,1,2,..,n$, there holds

$$\begin{aligned}&\psi\left(\left(\frac{b-a}{2}\right)^{m-i}\cdot\frac{1}{b-a}\cdot\int_a^b|u^{(m)}(x)|dx\right)\cdot\int_a^b|u^{(m)}(x)|dx\\&\leq(1+\beta)\cdot\max\left\{\left(\frac{b-a}{2}\right)^{\alpha(m-i)},\left(\frac{b-a}{2}\right)^{\beta(m-i)}\right\}\cdot\int_a^b\Psi(|u^{(m)}(x)|)dx.\end{aligned}$$

Proof. We only prove (i). Indeed, by Lemma 3 (i)-(iii), we get

$$\begin{aligned}&f_i\left(\left(\frac{b-a}{2}\right)^{m-i}\cdot\frac{1}{b-a}\cdot\int_a^b|u^{(m)}(x)|\mathrm{d}x\right)\cdot\frac{1}{b-a}\int_a^b|u^{(m)}(x)|\mathrm{d}x\\&\leq\max\left\{\left(\frac{b-a}{2}\right)^{\theta_i(m-i)},\left(\frac{b-a}{2}\right)^{\delta_i(m-i)}\right\}\cdot f_i\left(\frac{1}{b-a}\int_a^b|u^{(m)}(x)|\mathrm{d}x\right)\cdot\frac{1}{b-a}\int_a^b|u^{(m)}(x)|\mathrm{d}x\\&\leq\max\left\{\left(\frac{b-a}{2}\right)^{\theta_i(m-i)},\left(\frac{b-a}{2}\right)^{\delta_i(m-i)}\right\}\cdot(1+\delta_i)\cdot F_i\left(\frac{1}{b-a}\int_a^b|u^{(m)}(x)|\mathrm{d}x\right)\\&\leq\max\left\{\left(\frac{b-a}{2}\right)^{\theta_i(m-i)},\left(\frac{b-a}{2}\right)^{\delta_i(m-i)}\right\}\cdot(1+\delta_i)\cdot\frac{1}{b-a}\cdot\int_a^b F_i(|u^{(m)}(x)|)\mathrm{d}x.\end{aligned}$$

■

Proof of Theorem 1. Without loss of generality, we assume that $c=\int_a^b|u^{(m)}|\mathrm{d}x>0$. Otherwise, if $\int_a^b|u^{(m)}|\mathrm{d}x=0$, $|u^{(m)}|\equiv 0$ due to the continuity of $u^{(m)}$. That is to say $u(x)\equiv P_{m-1}(x)$, a polynomial of $(m-1)-$th degree. However, by the boundary conditions, it follows $P_{m-1}(x)\equiv 0$, which is a contradiction with our assumption that u is a nontrivial solution.

Firstly, we prove Theorem 1 (i) under the assumption that ψ satisfies the structural condition (H_ψ). Multiplying (4) by $u^{(m-1)}$, integrating over (a,b), and using Lemma 3 (iii), (H_2)-(H_4), Lemma 4, and Lemma 5 (ii), we get

$$\begin{aligned}
\int_a^b \Psi(|u^{(m)}|)\mathrm{d}x &\le \frac{1}{1+\alpha}\int_a^b \psi(|u^{(m)}|)|u^{(m)}|\mathrm{d}x\\
&= \frac{1}{1+\alpha}\int_a^b \psi(u^{(m)})u^{(m)}\mathrm{d}x\\
&= \frac{1}{1+\alpha}\sum_{i=0}^{n}\int_a^b r_i f_i(u^{(i)})u^{(m-1)}\mathrm{d}x\\
&\le \frac{1}{1+\alpha}\sum_{i=0}^{n}\left(\int_a^b |r_i|\mathrm{d}x\cdot \max_{x\in[a,b]}|f_i(u^{(i)})u^{(m-1)}|\right)\\
&\le \frac{1}{1+\alpha}\sum_{i=0}^{n}\left(\int_a^b |r_i|\mathrm{d}x\cdot \max_{x\in[a,b]}\left(k_i\psi(|u^{(i)}|)\cdot|u^{(m-1)}|\right)\right)\\
&\le \frac{1}{1+\alpha}\sum_{i=0}^{n}\left(\int_a^b k_i|r_i|\mathrm{d}x\cdot \psi\left(\frac{(b-a)^{m-i-1}}{2^{m-i}}\int_a^b|u^{(m)}|\mathrm{d}x\right)\cdot\frac{1}{2}\int_a^b|u^{(m)}|\mathrm{d}x\right)\\
&\le \frac{1+\beta}{2(1+\alpha)}\int_a^b \Psi(|u^{(m)}|)\mathrm{d}x\cdot\sum_{i=0}^{n}\left(\int_a^b k_i|r_i|\mathrm{d}x\cdot\max\left\{\left(\frac{b-a}{2}\right)^{\alpha(m-i)},\left(\frac{b-a}{2}\right)^{\beta(m-i)}\right\}\right).
\end{aligned}$$

Note that $\int_a^b \Psi(|u^{(m)}|)\mathrm{d}x > 0$. Otherwise, $\int_a^b \Psi(|u^{(m)}|)\mathrm{d}x = 0$. By Lemma 5 (ii) and Lemma 3 (i), we deduce that for $i=0,1,2,...,n$, there holds

$$0\le c\psi(t) = c\psi\left(\frac{2^{m-i}t}{c}\cdot\frac{c}{2^{m-i}}\right)\le \max\left\{\left(\frac{2^{m-i}t}{c}\right)^{\alpha},\left(\frac{2^{m-i}t}{c}\right)^{\beta}\right\}\psi\left(\frac{c}{2^{m-i}}\right)c\le 0,\ \ \forall t\ge 0,$$

which implies $\psi\equiv 0$ for all $t\in[0,+\infty)$. Then by the odd property of ψ, we have $\psi\equiv 0$ for all $t\in(-\infty,+\infty)$. Due to (H_4), $\sum\limits_{i=0}^{n}|f_i|\equiv 0$ for all $t\in(-\infty,+\infty)$, which is a contradiction with the assumption (H_1). Finally, the desired result can be obtained. Theorem 1 (i) has been proven.

Now we prove Theorem 1 (ii). Let $j_0\in[0,n]$ be a integer such that

$$I_{j_0} = \int_a^b \frac{1}{k_{j_0}}(1+\theta_{j_0})F_{j_0}(|u^{(m)}|)\mathrm{d}x = \max_{j=0,\dots,n}\left\{\int_a^b \frac{1}{k_j}(1+\theta_j)F_j(|u^{(m)}|)\mathrm{d}x\right\}.$$

Using Lemma 3 (iii), (H_2)-(H_4), Lemma 4 and Lemma 5 (i), we get

$$\begin{aligned}
I_{j_0} &\le \int_a^b \frac{1}{k_{j_0}}f_{j_0}(|u^{(m)}|)|u^{(m)}|\mathrm{d}x\\
&\le \int_a^b \psi(|u^{(m)}|)|u^{(m)}|\mathrm{d}x\\
&= \int_a^b \psi(u^{(m)})u^{(m)}\mathrm{d}x\\
&= \int_a^b \sum_{i=0}^{n} r_i f_i(u^{(i)})u^{(m-1)}\mathrm{d}x\\
&\le \sum_{i=0}^{n}\left(\max\{|f_i(u^{(i)})u^{(m-1)}|\}\cdot\int_a^b |r_i(x)|\mathrm{d}x\right)\\
&\le \sum_{i=0}^{n}\left(\int_a^b |r_i|\mathrm{d}x\cdot f_i\left(\frac{(b-a)^{m-i-1}}{2^{m-i}}\int_a^b|u^{(m)}|\mathrm{d}x\right)\cdot\frac{1}{2}\int_a^b|u^{(m)}|\mathrm{d}x\right)
\end{aligned}$$

$$\le \frac{1}{2}\sum_{i=0}^{n}\left(\int_a^b |r_i|\mathrm{d}x\cdot(1+\delta_i)\cdot\max\left\{\left(\frac{b-a}{2}\right)^{\theta_i(m-i)},\left(\frac{b-a}{2}\right)^{\delta_i(m-i)}\right\}\cdot\int_a^b F_i(|u^{(m)}|)\mathrm{d}x\right)$$
$$=\frac{1}{2}\sum_{i=0}^{n}\left(\int_a^b k_i|r_i|\mathrm{d}x\cdot\frac{1+\delta_i}{1+\theta_i}\cdot\max\left\{\left(\frac{b-a}{2}\right)^{\theta_i(m-i)},\left(\frac{b-a}{2}\right)^{\delta_i(m-i)}\right\}\cdot\int_a^b \frac{1+\theta_i}{k_i}F_i(|u^{(m)}|)\mathrm{d}x\right)$$
$$\le I_{j_0}\cdot\frac{1}{2}\sum_{i=0}^{n}\left(\int_a^b k_i|r_i|\mathrm{d}x\cdot\frac{1+\delta_i}{1+\theta_i}\cdot\max\left\{\left(\frac{b-a}{2}\right)^{\theta_i(m-i)},\left(\frac{b-a}{2}\right)^{\delta_i(m-i)}\right\}\right). \tag{5}$$

Note that , $k_{j_0}\ge 0, 1+\theta_{j_0}>0$ and $F_{j_0}(|u^{(m)}|)\ge 0$, arguing as in the proof of (i), we may get $I_{j_0}>0$. Finally, (5) implies the desired result. ■

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